Snyder-de Sitter model from two-time physics

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Abstract

We show that the symplectic structure of the Snyder model on a de Sitter background can be derived from two-time physics in seven dimensions and propose a Hamiltonian for a free particle consistent with the symmetries of the model.

1 Introduction

Some time ago, Romero and Zamora [1] derived the phase space of the Snyder model [2] from the two-time (2T) physics model introduced in ref. [3][1].

Two-time physics is a D-dimensional model defined on phase space, with two timelike dimensions, having the $SO(D-2,2)$ group as a global symmetry and the two-dimensional symplectic group $Sp(2)$ as the local one, whose Hamiltonian is given by a combination of homogeneous quadratic constraints in the phase space variables. The introduction of two coordinates with timelike signature is necessary in order to satisfy the constraints in a nontrivial way. By fixing the gauge freedom, one can recover several well-known four-dimensional models, like the massive relativistic particle in flat spacetime, or the massless particle in de Sitter spacetime. This formalism makes manifest some hidden symmetries of ordinary physics and can be extended to include background gauge fields of any spin and noncommutative spacetime [5].

[1]In a different context, a similar model was also discussed in [4].

As for the Snyder model, it was introduced long time ago, in order to show the possibility of defining a noncommutative spacetime invariant under the action of the Lorentz group. Later, it was interpreted as an instance of doubly special relativity [6]. This is a theory admitting two fundamental scales [7], usually identified with the speed of light and the Planck mass. The presence of two fundamental scales enforces a deformation of the spacetime symmetries, in particular of translation invariance [8].

More recently, the Snyder model has been generalized to the case of a de Sitter background [9, 10]. The resulting Snyder-de Sitter (SdS) model, called triply special relativity by the authors of ref. [9] because of the presence of the cosmological constant as a third independent fundamental scale, displays a duality between position and momentum coordinates.

Although the generalization of the Snyder model to a de Sitter background is not unique [10], a notably elegant extension has been proposed in ref. [9]. The fundamental Poisson brackets postulated in [9] read

$$\begin{aligned}\{X_\mu, X_\nu\} &= -\frac{1}{\kappa^2}(X_\mu P_\nu - X_\nu P_\mu),\\ \{P_\mu, P_\nu\} &= -\frac{1}{\alpha^2}(X_\mu P_\nu - X_\nu P_\mu),\\ \{X_\mu, P_\nu\} &= \eta_{\mu\nu} - \frac{1}{\alpha^2}X_\mu X_\nu - \frac{1}{\kappa^2}P_\mu P_\nu - \frac{2}{\alpha\kappa}P_\mu X_\nu, \end{aligned}\tag{1}$$

with κ the Planck energy and α the de Sitter radius. As discussed in [11], this specific realization of the SdS model turns out to be a nonlinear realization of the Yang model [12], proposed by Yang soon after Snyder's paper.

Previously, a derivation of the SdS model from a six-dimensional system with Lorentzian signature and nonhomogeneous constraints has been given in [13]. In this paper we provide an alternative derivation from a different higher-dimensional system, showing that the SdS model can be obtained starting from 7-dimensional two-time physics.

2 The 2T model

The 2T model [3] is defined on a flat D-dimensional manifold with two timelike coordinates and signature $(+,-,\dots,-,+)$. Its action can be written as[2] [3]

$$S = \int \left[\dot{X}\cdot P - \left(\lambda_1 \frac{1}{2}P^2 + \lambda_2 X\cdot P + \lambda_3 \frac{1}{2}X^2\right)\right] d\tau, \tag{2}$$

[2]Capital latin indices run from 0 to $D-1$. We denote $V^2 \equiv V_A V^A$, $V\cdot W \equiv V_A W^A$. In the following, greek indices run from 0 to 3 and $V_\rho^2 \equiv V^\rho V_\rho$.

and is invariant under the global $SO(D-2,2)$ symmetry with generators $J_{AB} \equiv X_A P_B - X_B P_A$. The Hamiltonian is therefore

$$H = \lambda_1 \frac{1}{2} P^2 + \lambda_2 X \cdot P + \lambda_3 \frac{1}{2} X^2, \tag{3}$$

with λ_1, λ_2 and λ_3 Lagrange multipliers that enforce the constraints

$$\begin{aligned} \phi_1 &= \frac{1}{2} P^2 \approx 0, \\ \phi_2 &= X \cdot P \approx 0, \\ \phi_3 &= \frac{1}{2} X^2 \approx 0, \end{aligned} \tag{4}$$

where the weak equivalence is used in the sense of Dirac [14]. The Hamilton equations following from (2) are

$$\begin{aligned} \dot{X}_A &= \{X_A, H\} = \lambda_1 P_A + \lambda_2 X_A, \\ \dot{P}_A &= \{P_A, H\} = -\lambda_2 P_A - \lambda_3 X_A. \end{aligned} \tag{5}$$

No secondary constraints are present, since

$$\begin{aligned} \dot{\phi}_1 &= \{\phi_1, H\} = -2\lambda_2 \phi_1 - \lambda_3 \phi_2 \approx 0, \\ \dot{\phi}_2 &= \{\phi_2, H\} = -2\lambda_1 \phi_1 - 2\lambda_3 \phi_3 \approx 0, \\ \dot{\phi}_3 &= \{\phi_3, H\} = \lambda_1 \phi_2 + 2\lambda_2 \phi_3 \approx 0. \end{aligned}$$

The Poisson brackets between the constraints generate the $sp(2)$ algebra,

$$\{\phi_1, \phi_2\} = -2\phi_1, \qquad \{\phi_1, \phi_3\} = -\phi_2, \qquad \{\phi_2, \phi_3\} = -2\phi_3, \tag{6}$$

and all the constraints are therefore first class. Due to the presence of three first-class constraints, the original $2D$ coordinates of the phase space reduce after gauge fixing to $2(D-3)$ independent ones.

3 Snyder space

In constrained Hamiltonian systems, the presence of arbitrary functions λ_i in the Hamiltonian indicates that the correspondence between the physical states and the canonical variables is not one to one, but to a given state can correspond different sets of values of the canonical variables. This redundance is called gauge invariance and the transformations that connect different sets of equivalent variables are generated by the first class constraints [14]. The problem can be solved by imposing

new constraints, called gauge conditions, that reduce the first class constraints to second class, and decrease the number of independent variables, restoring a one-to-one correspondence between the physical states and the independent phase space coordinates.

In our case, the choice of specific gauge conditions leads to different lower-dimensional models. In particular, the authors of [1] showed that for $D = 6$, the choice

$$P_4 = L = \text{const}, \qquad\qquad X_4 = 0, \tag{7}$$

reduces the dynamics to that of a four-dimensional particle with independent variables X_μ and P_μ, while X_5 and P_5 become functions of the other variables,

$$P_5 = \sqrt{L^2 - P_\rho^2}, \qquad X_5 = \frac{-P_\sigma X^\sigma}{\sqrt{L^2 - P_\rho^2}}. \tag{8}$$

After imposing the gauge constraints (7), two of the constraints (4) become second class. For consistency, the gauge choice must be preserved under the evolution of the system. This can be achieved if one imposes $\lambda_1 = \lambda_2 = 0$, which is in accordance with the fact that after the elimination of X_4, X_5, P_4 and P_5, one is left with 8 degrees of freedom and a single first-class (Hamiltonian) constraint, given by

$$H = \frac{1}{2}\left[\frac{(P_\sigma X^\sigma)^2}{L^2 - P_\rho^2} + X_\rho^2\right] = 0. \tag{9}$$

Some drawbacks are however present in the definition of the Hamiltonian constraint. First of all, it does not look very attractive, although it can be written in the equivalent form

$$\left(\eta^{\mu\nu} - \frac{X^\mu X^\nu}{X_\rho^2}\right) P_\mu P_\nu = L^2, \tag{10}$$

resembling that of a massive particle. Moreover, the equations obeyed by the independent variables X_μ and P_μ are

$$\dot{X}_\mu = 0, \qquad \dot{P}_\mu = -\lambda_3 X_\mu, \tag{11}$$

and do not seem to have a sensible physical interpretation.

One can easily derive the Dirac brackets satisfied by the phase space variables after the elimination of the second class constraints,

$$\begin{aligned}
\{X_\mu, X_\nu\}^* &= -\frac{1}{L^2}(X_\mu P_\nu - X_\nu P_\mu), \\
\{P_\mu, P_\nu\}^* &= 0, \\
\{X_\mu, P_\nu\}^* &= \eta_{\mu\nu} - \frac{1}{L^2} P_\mu P_\nu.
\end{aligned} \tag{12}$$

These are precisely the commutation relations of Snyder space [2]. Had one chosen the alternative gauge $P_5 = L$, $X_5 = 0$, one would have recovered instead the so-called anti-Snyder model, which obeys the same Poisson brackets, with $L^2 \to -L^2$.

By duality, a similar calculation with the gauge choice

$$X_4 = M = \text{const}, \qquad\qquad P_4 = 0, \tag{13}$$

gives rise to a model with Dirac brackets for the phase space variables identical to the fundamental Poisson brackets of a free massless particle in de Sitter spacetime [4],

$$\begin{aligned}
\{X_\mu, X_\nu\}^* &= 0, \\
\{P_\mu, P_\nu\}^* &= -\frac{1}{M^2}(X_\mu P_\nu - X_\nu P_\mu), \\
\{X_\mu, P_\nu\}^* &= \eta_{\mu\nu} - \frac{1}{M^2} X_\mu X_\nu,
\end{aligned} \tag{14}$$

In this case, consistency requires $\lambda_2 = \lambda_3 = 0$, and the Hamiltonian constraint reads

$$H = \frac{1}{2}\left(\eta^{\mu\nu} + \frac{X^\mu X^\nu}{M^2 - X_\rho^2}\right) P_\mu P_\nu = 0, \tag{15}$$

which is proportional to that of a massless particle in de Sitter spacetime with cosmological constant M, in stereographic coordinates. The equations obeyed by the independent variables are

$$\dot{X}_\mu = \lambda_1 P_\mu, \qquad \dot{P}_\mu = 0. \tag{16}$$

Also in this case, the Hamiltonian and the equations of motion are not the standard ones for de Sitter space.

Finally we notice that, in analogy with the previous case, the gauge choice $X_5 = M$, $P_5 = 0$, would lead to anti-de Sitter spacetime.

4 The SdS model

From the previous results, one may guess that the SdS model can be obtained from the two-time model in a similar way. However, it turns out that the Poisson brackets of the SdS model can only be obtained starting from $D = 7$. The fixing of all the gauge degrees of freedom will then lead to a 8-dimensional phase space, and a further Hamiltonian constraint must be imposed if one wants to describe the dynamics of the 4-dimensional SdS particle.

It is natural to consider the following gauge conditions:

$$P_4 = L = \text{const}, \qquad\qquad X_4 = M = \text{const}. \tag{17}$$

Moreover, as a third gauge condition, we choose

$$MP_5 + LX_5 = 0. \tag{18}$$

In this way, we have fixed all the gauge freedom, and the Lagrange multipliers λ_i must vanish for consistency with (5). From the constraints and the gauge conditions it follows that

$$\begin{aligned}
X_5 &= \pm M\sqrt{\frac{X_\mu^2 P_\nu^2 - (X^\mu P_\mu)^2 - (MP_\mu - LX_\nu)^2}{4L^2M^2 - (MP_\rho + LX_\rho)^2}}, \\
P_5 &= \mp L\sqrt{\frac{X_\mu^2 P_\nu^2 - (X^\mu P_\mu)^2 - (MP_\mu - LX_\nu)^2}{4L^2M^2 - (MP_\rho + LX_\rho)^2}}, \\
X_6 &= \pm\frac{2M^2L - LX_\mu^2 - MX^\mu P_\mu}{\sqrt{4L^2M^2 - (MP_\rho + LX_\rho)^2}}, \\
P_6 &= \pm\frac{2ML^2 - MP_\mu^2 - LX^\mu P_\mu}{\sqrt{4L^2M^2 - (MP_\rho + LX_\rho)^2}}.
\end{aligned} \tag{19}$$

In this way we have fully reduced the system to a 8-dimensional one, spanned by the coordinates X_μ and P_μ.

Let's now consider the constraints (4) together with the gauge constraints

$$\begin{aligned}
\chi_1 &= P_4 - L \approx 0, \\
\chi_2 &= X_4 - M \approx 0, \\
\chi_3 &= MP_5 + LX_5 \approx 0,
\end{aligned} \tag{20}$$

and calculate their Poisson brackets. We obtain

$$\begin{aligned}
&\{\chi_1,\chi_2\} = 1, \qquad \{\chi_1,\chi_3\} = 0, \qquad \{\chi_2,\chi_3\} = 0 \\
&\{\phi_1,\chi_1\} = 0, \qquad \{\phi_1,\chi_2\} = -P_4, \qquad \{\phi_1,\chi_3\} = -LP_5, \\
&\{\phi_2,\chi_1\} = P_4, \qquad \{\phi_2,\chi_2\} = -X_4, \qquad \{\phi_2,\chi_3\} = MP_5 - LX_5, \\
&\{\phi_3,\chi_1\} = X_4, \qquad \{\phi_3,\chi_2\} = 0, \qquad \{\phi_3,\chi_3\} = MX_5.
\end{aligned} \tag{21}$$

It follows that all constraints are now second class. Their Poisson brackets are

encoded in the following matrix:

$$C_{\alpha\beta} \equiv \{\chi_\alpha, \chi_\beta\} = \begin{pmatrix} 0 & 0 & 0 & 0 & -L & -LP_5 \\ 0 & 0 & 0 & L & -M & 2MP_5 \\ 0 & 0 & 0 & M & 0 & -M^2P_5/L \\ 0 & -L & -M & 0 & 1 & 0 \\ L & M & 0 & -1 & 0 & 0 \\ LP_5 & -2MP_5 & M^2P_5/L & 0 & 0 & 0 \end{pmatrix},$$

whose inverse is

$$C^{\alpha\beta} = \frac{1}{4L^2M^2} \begin{pmatrix} 0 & M^2 & 2LM & M^3 & 3LM^2 & LM^2/P_5 \\ -M^2 & 0 & L^2 & -LM^2 & ML^2 & -L^2M/P_5 \\ -2LM & -L^2 & 0 & -3ML^2 & -L^3 & L^3/P_5 \\ -M^3 & LM^2 & 3ML^2 & 0 & 0 & 0 \\ -3LM^2 & -ML^2 & L^3 & 0 & 0 & 0 \\ -LM^2/P_5 & L^2M/P_5 & -L^3/P_5 & 0 & 0 & 0 \end{pmatrix}.$$

The Dirac brackets for the phase space coordinates X_μ, P_μ, defined as $\{A,B\}^* = \{A,B\} - \{A,\chi_\alpha\}C^{\alpha\beta}\{\chi_\beta,B\}$, are then given by

$$\begin{aligned} \{X_\mu, X_\nu\}^* &= -\frac{1}{4L^2}(X_\mu P_\nu - X_\nu P_\mu), \\ \{P_\mu, P_\nu\}^* &= -\frac{1}{4M^2}(X_\mu P_\nu - X_\nu P_\mu), \\ \{X_\mu, P_\nu\}^* &= \eta_{\mu\nu} - \frac{1}{4M^2}X_\mu X_\nu - \frac{1}{4L^2}P_\mu P_\nu - \frac{1}{2LM}P_\mu X_\nu. \end{aligned} \tag{22}$$

These are identical to the Poisson brackets (1) for $M = \alpha/2$, $L = \kappa/2$.

In order to define the dynamics, one must now add a further constraint, which corresponds to the Hamiltonian constraint of the ordinary relativistic particle. This was not necessary in the case of the flat Snyder model because, due to the lower dimensionality, one constraint was left after the reduction from six to four dimensions.

The most natural choice is the quadratic Casimir invariant of the residual $SO(1,4)$ symmetry generated by $J_{\mu\nu}$ and $J_{\mu 6}$. This is given by

$$H_4 = J_{\mu\nu}^2 + 2J_{\mu 6}^2 = N^2, \tag{23}$$

where N is a constant, proportional to the mass of the particle. Using (19), one obtains more explicitly, modulo a constant factor,

$$H_4 = LM\,\frac{M^2P_\mu^2 + L^2X_\mu^2 - X_\mu^2P_\nu^2 + (X^\mu P_\mu)^2 - 2LMX^\mu P_\mu}{4L^2M^2 - M^2P_\mu^2 - L^2X_\mu^2 - 2LMX^\mu P_\mu}. \tag{24}$$

In spite of the ugly expression of the Hamiltonian, the Hamilton equation derived from (24) with the help of the brackets (22), take a very simple form

$$\begin{aligned}
\dot{X}_\mu &= -\frac{1}{2L}\left[(1-N^2)LX_\mu - (1+N^2)MP_\mu\right], \\
\dot{P}_\mu &= -\frac{1}{2M}\left[(1+N^2)LX_\mu + (1-N^2)MP_\mu\right].
\end{aligned} \tag{25}$$

In second order form, using

$$P_\mu = \frac{1}{1+N^2}\,\frac{L}{M}\,[2\dot{X}_\mu - (1-N^2)X_\mu], \tag{26}$$

they reduce to

$$\ddot{X}_\mu = -N^2\,X_\mu. \tag{27}$$

Hence, each position coordinate satisfies the equation of a harmonic oscillator (or a free particle in the massless case).

A different possibility is to choose the Hamiltonian constraint like in de Sitter space, as proposed in [9],

$$(4M^2 - X_\mu^2)P_\nu^2 + (X^\mu P_\mu)^2 = N^2. \tag{28}$$

In this case the Hamilton equations are

$$\dot{X}_\mu = \frac{2}{L^2}\left[4L^2M^2 - M^2P_\rho^2 - L^2X_\rho^2 - 2LMX^\rho P_\rho\right]P_\mu, \qquad \dot{P}_\mu = 0. \tag{29}$$

and the momentum P_μ is conserved, while the coordinates X_μ satisfy coupled first order equations. However, the Hamiltonian breaks the symmetry for the interchange of P_μ with X_μ and therefore looks less natural than (24).

5 Conclusions

We have shown that the phase space of the SdS model can be realized starting from the 7-dimensional 2T model. Contrary to the derivation of the flat space Snyder model, the Hamiltonian constraint is not included in the original constraints, but must be added by hand. This fact leaves a greater freedom in the choice of the dynamics, avoiding the problems found in ref. [1], and allowing the introduction of massive particles. In particular, it is possible to choose a Hamiltonian that preserves the duality invariance for the interchange of X_μ and P_μ.

A different derivation of the SdS model from higher dimensions was proposed in [13], starting from a 6-dimensional model with Lorentz signature and inhomogeneous constraints. The possibility of using a lower dimensionality in this case,

is due to the inhomogeneity of the constraints, that reduces the local symmetry group to $U(1)$ instead of $Sp(2)$. In that derivation, however, the values of α and κ are no longer free, but have to be fixed from the beginning.

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